Broken Relativistic Symmetry Groups, Toroidal Moments and Superconductivity in Magnetoelectric Crystals

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Abstract

A connection between creation of toroidal moments and breaking of the relativistic crystalline group associated to a given crystal, is presented in this paper. Indeed, if magnetoelectric effects exist, the interaction between electrons and elementary magnetic cells appears in such a way that the resulting local polarization and magnetization break the local relativistic crystalline symmetry. Therefore, Goldstone bosons, also associated to toroidal moments, are created and, as a consequence, corresponding toroidal phases in crystals. The list of the Shubnikov groups compatible with this kind of phases is given and possible consequences in superconductor theory in magnetoelectric crystals are examined.

KEY WORDS: *toroidal moments, relativistic crystalline symmetries, symmetry breaking, anyons.*

PACS 1990: *05.30.L – Anyons and parastatistics. 61.50.E – Crystal symmetry, models and space groups, crystalline systems and classes. 74.30 – Superconductivity, General properties. 75.20 – Diamagnetism and paramagnetism. 75.80 – Magnetomechanical and magnetoelectric effects, magnetostriction.*

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1 - Introduction

The aim of this paper is to present a possible link between *magnetoelectric effects* and *toroidal moments* via the concept of *relativistic crystalline groups* [1, 2, 3, 4, 5] and to see how this link can be applied to derive a possible origin of superconductivity in electric and magnetic crystals. These crystals are characterized by their magnetic groups which are subgroups of the *Shubnikov group* $O(3)1'$. Among the 122 magnetic groups, only 106 are compatible with the existence of a linear or quadratic magnetoelectric effect [6]. Let us recall that, in these groups, the time inversion $1'$ appears in addition to or in combination with orthogonal transformations of the Euclidean space.

In the present paper, we will consider relativistic symmetry group theory in crystals. Therefore, we need an extension from the Shubnikov group $O(3)1'$ to the group $O(1,3)$ in the Minkowski space. More particularly, our attention will be devoted to transformations of $O(1,3)$ leaving invariant polarization and magnetization vectors and generating a subgroup of the relativistic point group associated to the magnetic group G of a given crystal, namely the normalizer $N(G)$ of G in $O(1,3)$.

This subgroup may not be identified to the magnetic group if G leaves invariant a particular non-zero velocity vector, *i.e.*, if G' and $G \subseteq G'$ are isotropy groups of $O(1,3)$. If such a vec-

tor exists and $G' \neq G$, one strictly speaks about the *relativistic crystalline symmetry* G'. We can represent, as in figure 1, the list of the magnetic groups for which one can have in a crystal a spontaneous magnetization, a spontaneous polarization or an invariant non-zero velocity vector [15]. Thus, as it can be seen in the figure 1, only 31 magnetic groups are compatible with the existence of a relativistic crystalline symmetry (see the groups contained into the lowest circle). The invariant non-zero velocity vectors can be linked with toroidal moments from the point of view of magnetic symmetries as it has been already shown in previous papers [7, 8]. In that one, we will explain how toroidal moments might be generated from broken relativistic crystalline symmetries and give a few assumptions in order to construct a superconductivity model.

2 - Preliminaries

The toroidal moments $\mathbf{T}^{(\ell)}$ are polar tensors which change sign under time inversion, like velocity vectors $\vec{v}$ or current vectors $\vec{j}$ of electric charges. There are referred in toroidal phase transitions as the *order parameter* and in particularly, in the superdiamagnetism of superconductors or dielectric diamagnetic bodies containing densely packed atoms (*"agregates"*) [7, 8, 9, 10, 11]. In such systems, in presence of spontaneous currents, there may exist states for which the configuration of the associated currents has a *tore-shaped solenoid with a winding* and so a dipolar toroidal moment (see figure 2).

More generally, the toroidal moments appear as form factors in the multipolar Taylor expansion of the current vectors satisfying the four-dimensional conservation law

$$\partial_\mu j^\mu(\vec{x},t) = 0\,.$$

For a system with a dipolar toroidal moment $\vec{T}$, the current density equals

$$\vec{j} = curl(curl(\vec{T}))\,.$$

The vector $\vec{T}$ is equal itself to $\vec{x}\,\xi\,(\vec{x},t)$, where $\xi(\vec{x},t)$ is the second Helmotz potential. In case of static multipolar electric moments, this latter can be expanded up to a constant factor as [12, 13]

$$\xi \equiv \sum_{k,\ell,m} k^2\,\mathbf{F}^\ell_{k,m}(\vec{x})\,\mathbf{T}^\ell_m(-k^2,t)\,,$$

where the $\mathbf{F}^\ell_{k,m}(\vec{x})$ functions are the basis functions of the Helmotz equation. The toroidal moments components $\mathbf{T}^\ell_m(-k^2=0,t)$ are defined by

$$\mathbf{T}^\ell_m(0,t) = -\tfrac{\sqrt{\ell\pi}}{(2\ell+1)}\int r^{\ell+1}\Big[\Big\{\mathbf{Y}^{(\ell-1)*} + \tfrac{2\sqrt{\ell}}{(2\ell+3)\sqrt{\ell+1}}\,\mathbf{Y}^{(\ell+1)*}\Big]\otimes\mathbf{j}^1(\vec{x},t)\Big\}^\ell_m\,d^3x\,,$$

where $r = \|\vec{x}\|$, $\mathbf{Y}^{(\ell)}$ is the well-known spherical harmonic of order ℓ, $\mathbf{j}^1 \equiv \vec{j}$ is the current vector expressed in the spherical basis and $\otimes$ indicates the tensor product of two irreducible tensor operators. For $\ell = 1$, we obtain the toroidal dipole moment of a given configuration of currents, and $\mathbf{T}^1_m$ can be expressed in a cartesian basis as follows

$$\mathbf{T}^1 \equiv \vec{T} = \frac{1}{10}\int\Big\{\vec{x}\,(\vec{x}\,.\,\vec{j}) - r^2\,\vec{j}\Big\}\,d^3x\,.$$

It has been shown [7, 8, 9, 10, 11] that, without any other modifications, one can formally substitute $\vec{T}$ for $\vec{v}$ in figure 1. We can remark that this kind of configuration of currents can be created from a solenoidal configuration of currents. The closure of the latter one can be obtain applying an external magnetic field, but we will present another possibility.

Hence, the problem is especially to determine the origin of, from one hand, the permanent solenoidal currents and, on the other, the permanent closure of these latter. The occurence of such phenomena is precisely discussed in connection with the breaking of the relativistic crystalline symmetries.

3 - Magnetoelectric Effects and Relativistic Symmetries

The state of a crystal is described with a one-valued function f depending on the electric, magnetic or/and constraint fields. Generally, f is taken to be the free enthalpy. The various properties of the crystal manifest themselves in the appearance of various tensors as coefficients, in the Taylor expansion of the function f, in terms of the different fields [2, 6]:

$$f = {}^oP_i\,E^i + {}^oM_i\,H^i + \tfrac{1}{2}\left(\epsilon_o\,\epsilon_{ik}\,E^i\,E^k + \mu_o\,\mu_{ik}\,H^i\,H^k\right) + \tfrac{1}{c}\,\alpha_o\,\alpha_{ik}\,E^i\,H^k + \cdots\,,$$

where ${}^oP$ and ${}^oM$ are respectively the electric and magnetic spontaneous polarizations, ϵ_{ik} and

μ_{ik}, the electric permitivity and magnetic permeability, and α_{ik}, the magnetoelectric succeptibility. Let us define the electric polarization component P_i and the magnetic polarization component M_i *via*

$$P_i \equiv \frac{\partial f}{\partial E^i} = {}^oP_i + \epsilon_o \epsilon_{ik}\, E^k + \frac{1}{c}\,\alpha_o \alpha_{ik}\, H^k + \cdots\,,$$

and

$$M_i \equiv \frac{\partial f}{\partial H^i} = {}^oM_i + \mu_o \mu_{ik}\, H^k + \frac{1}{c}\,\alpha_o \alpha_{ik}\, E^k + \cdots\,.$$

A tensor is different from zero if it remains invariant under all the symmetry transformations of the magnetic group of the crystal. The relativistic invariance of the non-zero permitted tensors of the Taylor expension has to be imposed because of the relativistic covariance of the Maxwell equations determining the dynamics of the electric and magnetic fields. Hence, a supplementary invariance is considered in addition to all the previous ones. Consequently, one applies the relativistic invariance to the second order term of the expension of f. It can be written [4] in the following form

$$f^{(2)} = \frac{1}{8c}\chi^{(\alpha\beta)(\gamma\delta)}\,\mathbf{F}_{\alpha\beta}\,\mathbf{F}_{\gamma\delta}\,,$$

where $\mathbf{F}$ is the Faraday tensor of the applied electromagnetic field and χ the relativistic succeptibility tensor.

We can represent the pairs of indices (α,β) and (γ,δ) by a single index

$$\left\{\begin{array}{l} (0,1) \longrightarrow 1\,,\\ (0,2) \longrightarrow 2\,,\\ (0,3) \longrightarrow 3\,,\\ (2,3) \longrightarrow 4\,,\\ (3,1) \longrightarrow 5\,,\\ (1,2) \longrightarrow 6\,. \end{array}\right.$$

Then, χ can be represented by a 6×6 matrix:

$$\chi^{(\alpha\beta)(\gamma\delta)} \equiv \sqrt{\frac{\epsilon_o}{\mu_o}}\left(\begin{array}{cc} -\epsilon^B & \alpha\,\mu^{-1} \\ {}^t(\alpha\,\mu^{-1}) & -\xi\,\mu^{-1} \end{array}\right),$$

with $\epsilon^B = \epsilon - \alpha\,\mu^{-1}\,{}^t\alpha$ (t as transposition) and where ϵ, μ, α and ξ are respectively the electric succeptibility tensor, the magnetic permeability tensor, the magnetoelectric succeptibility tensor and the magnetic succeptibility tensor.

The relativistic invariance imposes constraints (*i.e.* algebraic expressions) between the components of the tensors ϵ, μ, α and ξ. But before establishing these expressions, the relativistic crystalline group G' has to be determined. The definition of G' is the following: G' is the maximal group satisfying the relations [4]

$$G' \subseteq N(G) \cap K(\mathbf{P}_{e.m.})$$

and

$$G = G' \cap O(3)1'\,,$$

where G is the magnetic group of the crystal, $K(\mathbf{P}_{e.m.})$ the relativistic point group leaving invariant the polarization tensor $\mathbf{P}_{e.m.}$ of the crystal:

$$\mathbf{P}_{e.m.} \equiv \mathbf{P}^{\alpha\beta}_{e.m.} = \frac{1}{2}\chi^{(\alpha\beta)(\gamma\delta)}\mathbf{F}_{\gamma\delta}\,,$$

and $N(G)$ (*i.e.* the relativistic point group associated to G) the normalizer of G in $O(1,3)$, *i.e.*:

$$N(G) = \left\{ g \in O(1,3) \,/\, g\,G\,g^{-1} \subseteq G \right\}\,.$$

In fact in these kinds of definitions, one can also consider the polarization tensor $\mathbf{P}_{e.m.}$ not only defined as the polarization tensor of the whole crystal, but also from a more locally viewpoint, as the polarization tensor of an *"elementary magnetic cell"* of the crystal. We will consider this definition in all the further sections.

There are two cases of groups $K(\mathbf{P}_{e.m.})$ which are particularly interesting for the following discussion, *viz.*, K_M and $K_\angle(a,b)$, when using Asher notations [4]. These groups leave invariant the following polarization vector $\vec{P}$ and magnetization vector $\vec{M}$:

$$K_M: \quad \vec{M}(\neq\vec{0})//O_z \quad \text{and} \quad \vec{P} = \vec{0}\,,$$

$$K_\angle(a,b): \left\{\begin{array}{l} \vec{M}.\vec{P} \neq 0 \quad \text{and} \quad \vec{M}.\vec{P} \neq \|\vec{M}\|.\|\vec{P}\|\,,\\ \vec{P}//O_z \quad \text{and} \quad \vec{M} \perp O_y\,,\\ \text{with } a = c\,|M_x|/\|\vec{P}\|\,,\\ b = c\,|M_z|/\|\vec{P}\|\,,\\ \text{and } a\,b \neq 0\,. \end{array}\right.$$

The generators of K_M and $K_\angle(a,b)$ are

$$K_M \equiv \left\{ m'_y\,,\, \bar{1}\,,\, R_z(\phi)\,,\, L_z(\chi) \,/\, \forall\,\phi\,,\, \chi \in \mathbb{R} \right\}\,,$$

$$K_\angle(a,b) \equiv \left\{ m'_y\,,\, L(\phi;a,b)\,,\, \overline{L}(\chi;a,b) \,/\, \forall\,\phi,\chi \in \mathbb{R} \right\},$$

where $R_z(\phi)$ is a rotation of angle ϕ around the z axis, $L_z(\chi)$ is a special transformation with velocity βc in the z direction[1] and

$$L(\phi;a,b) = S^{-1}(a,b)\, R_z\,(\phi) S(a,b)\,, \qquad (1)$$

$$\overline{L}(\chi;a,b) = S^{-1}(a,b) L_z(\chi) S(a,b)\,,$$

where

$$S(a,b) = \begin{pmatrix} C & 0 & -S & 0 \\ 0 & c & 0 & -s \\ -S & 0 & C & 0 \\ 0 & s & 0 & c \end{pmatrix},$$

with $S(a,b) \in SO_y(1,1) \times SO_y(2)$,

$$\begin{array}{ll} C = \cosh(\xi)\,, & S = \sinh(\xi)\,, \\ c = \cos(\omega)\,, & s = \sin(\omega)\,, \end{array}$$

and ξ, ω are functions of a and b. Then, the procedure for establishing the constraints on the components of the tensor χ is the following: considering that $G = m'_y$ for example, then the normalizer $N(G)$ of m'_y in $O(1,3)$ is [4]

$$N(G) = O_y(1,1) \times O_y(2)\,.$$

We deduce that if $K(\mathbf{P}_{e.m.}) = K_M$ then the generators of the relativistic crystalline group G'_M are

$$G'_M \equiv \left\{ m'_y\,,\, R_z(\pi) \right\}\,,$$

whereas if $K(\mathbf{P}_{e.m.}) = K_\angle(a,b)$:

$$G'_\angle(a,b) \equiv \left\{ m'_y\,,\, L(\pi;a,b) \right\}\,.$$

The constraints on the tensor χ obviously derive from the relations $(\forall\, h \in G')$

$$\chi^{(\alpha\beta)(\gamma\delta)} = h^\alpha_\mu\, h^\beta_\nu\, h^\gamma_\eta\, h^\delta_\rho\, \chi^{(\mu\nu)(\eta\rho)}\,.$$

Clearly, these constraints change in function of G' itself, absolutly depending on the polarization and magnetization vectors. Thus, if for example the system passes from $\vec{P} = \vec{0}$ to $\vec{P} \neq \vec{0}$, it involves naturally a *"breaking of the relativistic crystalline symmetry"*. We must pay attention to this definition of the breaking of a symmetry, since usually a breaking involves a reduction to a proper subgroup of a given one, whereas in this case G'_M and $G'_\angle(a,b)$ are conjugate groups in $O(1,3)$ (see relation (1)). But nevertheless it can be considered as a breaking because the conjugate group is not a subgroup and the intersection of the two is a proper one of the broken group.

[1] $\chi = \cosh^{-1}([1/(1-\beta^2)]^{\frac{1}{2}}) = \cosh^{-1}(\gamma)\,.$

4- Relativistic broken symmetries and toroidal moments

We can summarize the previous results as follows.

By defining:

$$\bar{1}\, L_y(a,b) = S^{-1}(a,b)\, \bar{1}\, S(a,b)\,,$$

we have

$$\left\{ \begin{array}{l} S(a,b) \;\in\; N(G)\,, \\ \\ K^S_M = S(a,b)^{-1}\, K_M\, S(a,b) \\ \qquad = K_\angle(a,b)\, \bar{1}\, L_y(a,b)\,, \\ \\ N(G) \cap \{\bar{1}\, L_y(a,b)\} \;=\; \{1\}\,, \\ \\ K_\angle(a,b) \triangleleft K^S_M\,, \end{array} \right.$$

and

$$G'_M = S(a,b)\, G'_\angle(a,b)\, S^{-1}(a,b)\,. \qquad (2)$$

These results can be generalized in the case of a group K_M corresponding to a magnetization vector $\vec{M}$ in the perpendicular plane of the z-axis and not necessary parallel to O_z.

Then, we would have (O_M : $\vec{M}$-axis)

$$G'_M(\theta) = \{m'_y\,,\, R_{O_M}(\pi)\} = R_y(\theta)\, G'_M\, R^{-1}_y(\theta)\,,$$

where θ is the angle between $\vec{M}$ and O_z.

Obviously, the relation (2) would be analogous, replacing $S(a,b)$ by $R_y(\theta)S(a,b) = S(a,b,\theta) = S(a',b')$. It follows also from this relation (2) that the broken symmetry we generate, is again associated to the concept of conjugaison in the group $O(1,3)$ with respect to the connected component of the $N(G)$ group. Then, we consider the following classical physical system: $G'_M(\theta)$ is the relativistic crystalline group of an elementary polarized cell of the crystal, with an electron not in interaction with this cell in the initial state. We also assume that this electron has initially (before intraction) a Galilean motion with a velocity $\vec{v}$ parallel to O_y and that the axis of motion is at the distance $x_o \neq 0$ of the y-axis. Then, the electron interacts with the cell, inducing (by a *"local magneto-electric effect"*), an electric polarization, breaking the initial relativistic crystalline symmetry group $G'_M(\theta)$.

During the interaction, the resulting symmetry group is $G'_\angle(a',b')$. The transformation from one symmetry to the other is realized with the

matrix $S(a',b') \in SO_y(1,1) \times SO_y(2)$. It means that we pass with $S(a',b')$, from one frame to another in which the motion of the electron is kept, *i.e.*, Galilean. In some way, we can speak of a kind of *"inverse kinetomagnetoelectric effect"* [15]. Thus, the motion of the electron in the laboratory frame is determined by $S(a',b')$, *i.e.*, a rotation around the y-axis and a boost along the same axis. From the latter characteristics of the motion of the electron and those of $S(a',b')$, the electron will possess a solenoidal motion around the y-axis.

Then, considering similarly another cell and an electron moving with a fast solenoidal motion along an axis parallel to the y-axis, the interaction with the cell might provide a closure of the solenoidal configuration into a toroidal configuration in case of an adiabatic process. Let us precise this dynamic from a more mathematical viewpoint in the case of a "classical motion" only, since this latter is phenomenologically closed to the quantum one.

5 - The evolution equation

We have precised the meaning of the transformation S: it equivalently defines a change of Galilean frame during the electron-cell interaction. We will use this *"equivalence principle"* (*) to obtain the type of motion or trajectories of the interacting electrons. We consider that passing from a non-interacting electron-cell system to an interacting one, is equivalent to pass from a Galilean frame $\mathbf{R}_t$ to another $\mathbf{R'}_{t'}$, with the transformation S keeping in these two frames, the direction of a velocity vector $\vec{v}$ invariant, irrespectively of its module.

The coordinates of the position vector of the electron will be $\widetilde{X} = (t, x, y, z)$ in $\mathbf{R}_t$, and $\widetilde{X}' = (t', x', y', z')$ in $\mathbf{R'}_{t'}$.

It follows that

$$\widetilde{X}' = S\,\widetilde{X}\,, \tag{3}$$

where $S \in N(G)$ and S function of t' (or t).

At this stage, if $\widetilde{X}$ and $\widetilde{X}'$ have a variation due to an infinitesimal change of the $\mathbf{R}_t$ and $\mathbf{R'}_{t'}$ frames, then in order to keep the relativistic covariance of the relation (3), this transformation has to be an infinitesimal Lorentz transformation in $N(G)$. Hence

$$\begin{aligned} \widetilde{X}' + d\widetilde{X}' &= (\mathbf{1} - dn')\widetilde{X}'\,, \\ \widetilde{X} + d\widetilde{X} &= (\mathbf{1} - dn)\widetilde{X}\,, \end{aligned} \tag{4}$$

where dn and dn' are elements of the Lie algebra of $N(G)$. Then, from equation (3), we deduce that

$$d\widetilde{X}' = (dS)\widetilde{X} \;+\; S\,d\widetilde{X} = ((dS)S^{-1})\widetilde{X}' + S\,d\widetilde{X}\,,$$

and from equation (4)

$$d\widetilde{X}' = ((dS)S^{-1})\widetilde{X}' - (S\,dn\,S^{-1})\widetilde{X}'\,,$$

$$d\widetilde{X}' = ((dS)S^{-1} - S\,dn\,S^{-1})\widetilde{X}' = -dn'\widetilde{X}'\,. \tag{5}$$

Now, if we ascribe the variations of $\widetilde{X}$ and $\widetilde{X}'$ to the determination of these two 4-vectors in a third frame $\mathbf{R''}_{t''} \equiv \mathbf{R'}_{t'+dt'}$ obtained from $\mathbf{R'}_{t'}$ by an infinitesimal evolution in time: dt', of this latter frame, then equation (5) can be written as

$$\frac{d\widetilde{X}'}{dt'} = \left(\left(\frac{dS}{dt'}\right)S^{-1} - S\left(\frac{dn}{dt'}\right)S^{-1}\right)\widetilde{X}'\,.$$

Of course, we recognize a gauge transformation with respect to the $N(G)$ group of an affine connexion dn/dt' of the frames bundle of the Minkowski space, since we have the well-known relation in gauge theory

$$dn' = S\,dn\,S^{-1} - (dS)\,S^{-1}\,.$$

More, we recognize particularly a kind of *Thomas precession equation* [17, 18] which explains the solenoidal motion of the electron during the interaction with the magnetic cell. Let us add that in accordance with the equivalence principle (*) we gave, the $\mathbf{R''}_{t''}$ frame can be associated to an evolution of the electron-cell interaction (or of the breaking of the relativisic crystalline symmetry of the cell). If for instance, S is of the form

$$S = \begin{pmatrix} \cosh(\xi t') & 0 & -\sinh(\xi t') & 0 \\ 0 & \cos(\omega t') & 0 & -\sin(\omega t') \\ -\sinh(\xi t') & 0 & \cosh(\xi t') & 0 \\ 0 & \sin(\omega t') & 0 & \cos(\omega t') \end{pmatrix}$$

and $\widetilde{X} = (ct, x_o, vt, 0)$, where $v = \|\vec{v}\|$ is the module of the invariant velocity vector $\vec{v}$, we obtain

$$\begin{cases} ct' &= ct\cosh(\xi t') - vt\sinh(\xi t')\,, \\ x'(t') &= x_o\cos(\omega t')\,, \\ y'(t') &= vt\cosh(\xi t') - ct\sinh(\xi t')\,, \\ z'(t') &= x_o\sin(\omega t')\,. \end{cases}$$

Now, because of the equivalence principle (*), the relations above must be satisfied whatever

is the module v. Then, we have necessarily $\xi = 0$ and the time t' has to be equaling to the time t of the laboratory frame. Then $y'(t) = vt$ and $x'(t)^2 + z'(t)^2 = 1$. Hence, the electron has a solenoidal motion as expected. In conclusion, only $SO(2)$ acts on the dynamic of the electron because of the equivalence between t and t', whereas $N(G)$ is the gauge group of the electron-cell system. To conclude, the motion of the electron is invariant with respect to the maximal group contained in $N(G) \cap O(3)$. Let us remark that the inclusion in $O(3)$ is a well-known result of the precession theory.

In the case of a polarized electron, the Thomas precession of its spin along the trajectory would lead to a toroidal configuration of spin currents [16]. On the other hand, if the electron has a fast solenoidal motion before the interaction with the cell, then with the condition that the interaction is adiabatic, one will obtain, by the latter process, a slow solenoidal motion of the mean position of the electron, and so a toroidal motion during the interaction.

6 - The compatible Shubnikov groups

In the case of the m'_y symmetry, such toroidal motions cannot be completly compatible with this symmetry because they produce a non-vanishing orbital moment along the y-axis. Then, in order to have a non-zero toroidal moment, it is necessary to assume the existence of a second electron coupled with the first one with the same toroidal moment but with an opposed orbital moment. In this case the crystal will be in a kind of "ferro-toroidal" phase (T^2) (the exponent 2 for two electrons).

One can generalize the latter procedure to other magnetic groups. The groups G for which toroidal moments may exist, must possess a normalizer $N(G)$ containing the group $O(2)$. From the classification of the normalizers given by E. Asher, one can state that the only (fifteen groups plus the group 1) compatible magnetic groups are [4]

$$\begin{array}{cccccccc} 1, & \bar{1}', & 2, & 2', & 3, & 3', & 4, & 4', \\ 6, & 6', & m, & m', & 2/m', & 2'/m, & 4/m', & 6/m'. \end{array}$$

Then, applying the latter tools, we can work out the toroidal phases for each magnetic group, knowing the possible orientations of the magnetic, electric and kinetic polarization and the permitted $K(\mathbf{P}_{e.m.})$ groups. The results are given in Table I, following the H. Schmid classification and notations [6].

In order to illustrate how to construct the Table I, one can give two examples: the group m, and a AA-type one. In the case of m, the orientations of $\vec{P}$, $\vec{M}$ and $\vec{v}$ are represented in figure 3. The normalizer $N(m)$ is: $O_{xz}(1,2)\dot{\times}m_y$. The possible basic continuous transformations from one polarization state to another one are ($S(a)$: boost, $R_y(\theta)$: rotation around the y-axis with an angle θ. The null polarization and magnetization vectors are not indicated):

$$\left\{ \begin{array}{lcl} \vec{P} & \longrightarrow & \vec{P}' \quad \text{and} \quad \widehat{(\vec{P},\vec{P}')} = \theta\,, \\ K_P & \longrightarrow & K_\perp(a,\theta) = \\ & & R_y^{-1}(\theta)S^{-1}(a)K_P S(a)R_y(\theta)\,, \end{array} \right.$$

$$\left\{ \begin{array}{lcl} \vec{M} & \longrightarrow & \vec{P} \quad \text{and} \quad \vec{M}'//\vec{M}\,, \\ K_M & \longrightarrow & K_\perp(a) = S^{-1}(a)K_M S(a)\,, \end{array} \right.$$

$$\left\{ \begin{array}{lcl} \vec{M} & \longrightarrow & \vec{M}'//\vec{M}\,, \\ K_M & \longrightarrow & K_{M'} = K_M\,, \end{array} \right.$$

$$\left\{ \begin{array}{lcl} \vec{P} & \longrightarrow & \vec{P}' \quad \text{and} \quad \widehat{(\vec{P},\vec{P}')} = \theta\,, \\ K_P & \longrightarrow & K_{P'}(\theta) = R_y^{-1}(\theta)K_P R_y(\theta)\,. \end{array} \right.$$

All the other transformations can be deduced from these four. The computation of the groups G' in each case shows that no conjugaisons are possible between two groups G' with a rotation around an axis contained in the plane m. It follows that no broken symmetry appears and consequently no toroidal moments contained in the plane m. Only conjugaisons with $R_y(\theta)$ can be taken, which may suggest the existence of kind of "antiferro-toroidal phases" along the y-axis ($\overline{T}_y$) (Two opposite toroidal moments per magnetic cell along the y-axis). In the case of G groups of type AA, any orientations in each sublattices of the crystal are permitted so that toroidal moments may appear if oblique fields exist since in this case, it is always possible to conjugate by a rotation two $K(\mathbf{P}_{e.m.})$ groups. As a result, making the same kind of computation as above, we find that only 12 groups among the 16 ones tabulated in Table I are associated to a non-trivial $SO(2)$ action, and then can induced a creation of a solenoidal motion, *i.e.* anyons.

From these results, one can do few hypothesis as conclusion about a model of superconductivity in magnetoelectric crystals.

7 - Conclusion

We have seen in the previous sections that the breaking of relativistic symmetry can produce a toroidal moment. But in the same time, the electron, interacting with the cell, acquires an orbital moment, whereas, in the initial state, the orbital moment of the electron-cell system was zero. As a result, a quasi-particle is produced during the interaction with a non-zero orbital moment in order to keep the total orbital moment. This quasi-particle is associated with a broken symmetry. Strictly speaking, it is a Goldstone boson. The latter one is connected with the propagation in the crystal of the broken symmetry, *i.e.*, the variation of the local polarization and magnetization resulting from a magnetoelectric interaction between cells and electrons. Consequently, it will carried an electric and magnetic moment and a non-vanishing orbital moment.

Then, one can consider a superconductivity model fitting the B.C.S. theory where the Goldstone bosons, and not the phonons, will couple the electrons of the Cooper pairs. Models of the electron-hole pairing involving toroidal moments have been already studied in the case of excitonic-insulators [13]. Let us mention that the Goldstone bosons will interact between themselves via their electric and magnetic moments, *i.e.*, via segnetomagnon modes. Obviously, such bosons can be directly responsible for the coupling of two toroidal moments in the T^2 phase for example.

We can precise the characteristics of the Goldstone bosons considering the cyon theory [19, 20, 23]. Indeed, the cyons are types of atoms with one electron and no nucleus in which the electron is constrained to move in a toroidal domain of the space. The cyon theory has been linked with another one: the dyon theory [19, 20, 23]. The dyons are made of one electron and a Dirac monopole as nucleus of the atom. In the model described in this paper, in order to explain the origin of toroidal moments, we considered an electron with a solenoidal motion in the initial state.

Bearing in mind that the potential vector $\widetilde{A}$ of a Dirac monopole can be generated by a semi-infinite solenoid of electrical currents along the semi-infinite singularity axis of the potential vector, we can consider that the interaction of the electron with the cell will produce a dyon in which the single electron would have an orbital with a toroidal configuration. Hence, we would obtain a cyon and a Dirac monopole. Let us note that a link exists between toroidal moments and magnetic charges, as it has been already shown [13]. The toroidal configuration will be associated with an invariance of the electron-cell system with respect to the group $N(G)$ only (containing $SO(2)$ as a gauge group of the dyon and of the potential vector of the monopole). Then, it follows that we may consider the Goldstone bosons associated with the breaking of the relativistic symmetry as an effective Dirac monopole. We can also precise that we assumed an adiabatic electron-cell interaction so that a Berry phase appears from a Yang-Mills gauge field in the equivalent quantum system. It has been shown that this gauge field can be associated to an anomaly such as a monopole for example.

If we turn out to Lagrangian formalism, the latter remark means that beginning with a Lagrangian describing the electrons and invariant with respect to the space group of the crystal, we will consider the interaction with the cell in the crystal via magnetoelectric effects, adding perturbative Lagrangians coming from a gauge interaction associated to the gauge group $N(G)$. This group will break the relativistic crystalline symmetry of the cell during the interaction. The resulting electromagnetic gauge field will be associated to the effective Dirac monopole in such a way that the total Lagrangian be invariant only with respect to the group $N(G)$ (the connected component of $N(G)$ being the gauge group). Furthermore, the question arises to know if two electrons with a toroidal configuration of currents, and consequently producing a toroidal moment, may described anyons and the breakdown of time-reversal (T) and parity (P) symmetries [21, 22]. Indeed, toroidal moments break these two symmetries but not the PT one as expected in anyon theory. More, cyons are anyons, and the flux tube of anyons can be generated by a monopole [23].

From an experimental viewpoint, it would be necessary to characterize the toroidal moments by interaction with light or hyperfrequency electric or/and magnetic fields. Absorption of far infra-red light under static magnetic fields have been applied and measurements of the circular dichroism are now available [9, 13]. To close, let us note that under highfrequency fields the crystal would present a diamagnetic-paramagnetic transition [13].

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Figure captions

Table I:

⊙ : "Weak ferromagnetism" permitted (I. F. Dzyaloshinskii, (1957) [14]).

† : "Weak ferroelectricity" permitted.

Type of ordering:

- M = pyro-, ferro- or ferrimagnetic;
  P = pyro-, ferro- or ferrielectric.
- $\overline{M}$ = antiferromagnetic;
  $\overline{P}$ = antiferroelectric or orthoelectric.
- T = pyro-, ferro-, or ferritoroidal;
  T^2 = same phases as T but with two electrons per toroidal moment.
- $\overline{T}$ = antiferrotoroidal phase;
  $\overline{T}^2$ = same phases as $\overline{T}$ but with two electrons per toroidal moment.
- $\overline{T}_y$ = antiferrotoroidal phase along the y-axis.

- H: spontaneous magnetization permitted; E: spontaneous polarization permitted.
- EH: linear magnetoelectric effect permitted.
- EEH and HEE: second-order magnetoelectric effect.
- P^S: invariant electric polarization vector; M^S: invariant magnetization vector; T^S: invariant toroidal vector; v: velocity vector.
- $+$: meaning all directions permitted.
- $\dot{\times}$: semi-direct product.
- (y): perpendicular to the y-axis, y: along the y-axis.
- m_v: symmetry with respect to the v-plane.

Figure 2:
Configuration of a current having a toroidal dipole moment. The arrows on the torus indicate the current direction, and the moment is directed along the symmetry axis of the torus.

Figure 3:
Orientations of the polarization vectors P and M, and the velocity vector v in case of a m symmetry.

Magneto-electric type	Permitted terms of stored free enthalpie	Type of ordering		Shubnikov point groups	P^S in cell [2]	M^S in cell [2]	T^S (or v) in cell [2]	Normalizer [4]
		Magnetic	Electric					
FEI/FMI	E H ⊙EH HEE EHH	M	P	T, T^2	$\odot^\dagger 1$	$+$	$+$	$O(1,3)$
				—	$\odot^\dagger 2,\ 3,\ 4,\ 6$	y	y	$O(1,1)\times O(2)$
				$\overline{T}_y$	$\odot^\dagger 2'$	(y)	(y)(for v only)	$O(1,2)\dot{\times} m_v$
				$\overline{T}_y$	$\odot^\dagger m$	y	(y)(for v only)	$O(1,2)\dot{\times} m_v$
				T^2	$\odot^\dagger m'$	(y)	y	$O(1,1)\times O(2)$
AAIII	⊙EH	$\overline{M}$	$\overline{P}$	$T, \overline{T}^2$	$\odot\bar{1}'$	$+$	$+$	$O(1,3)$
					$2'/m$	$+$	$+$	$O(1,2)\dot{\times} m_v$
					$\bar{3}',\ 2/m'$ $4/m',\ 6/m'$	$+$	$+$	$O(1,1)\times O(2)$
AAI	HEE ⊙EH HHE	$\overline{M}$	$\overline{P}$	$\overline{T}, \overline{T}^2$	$\bar{4}',\ \bar{6}'$	$+$	$+$	$O(1,1)\times O(2)$

Table I

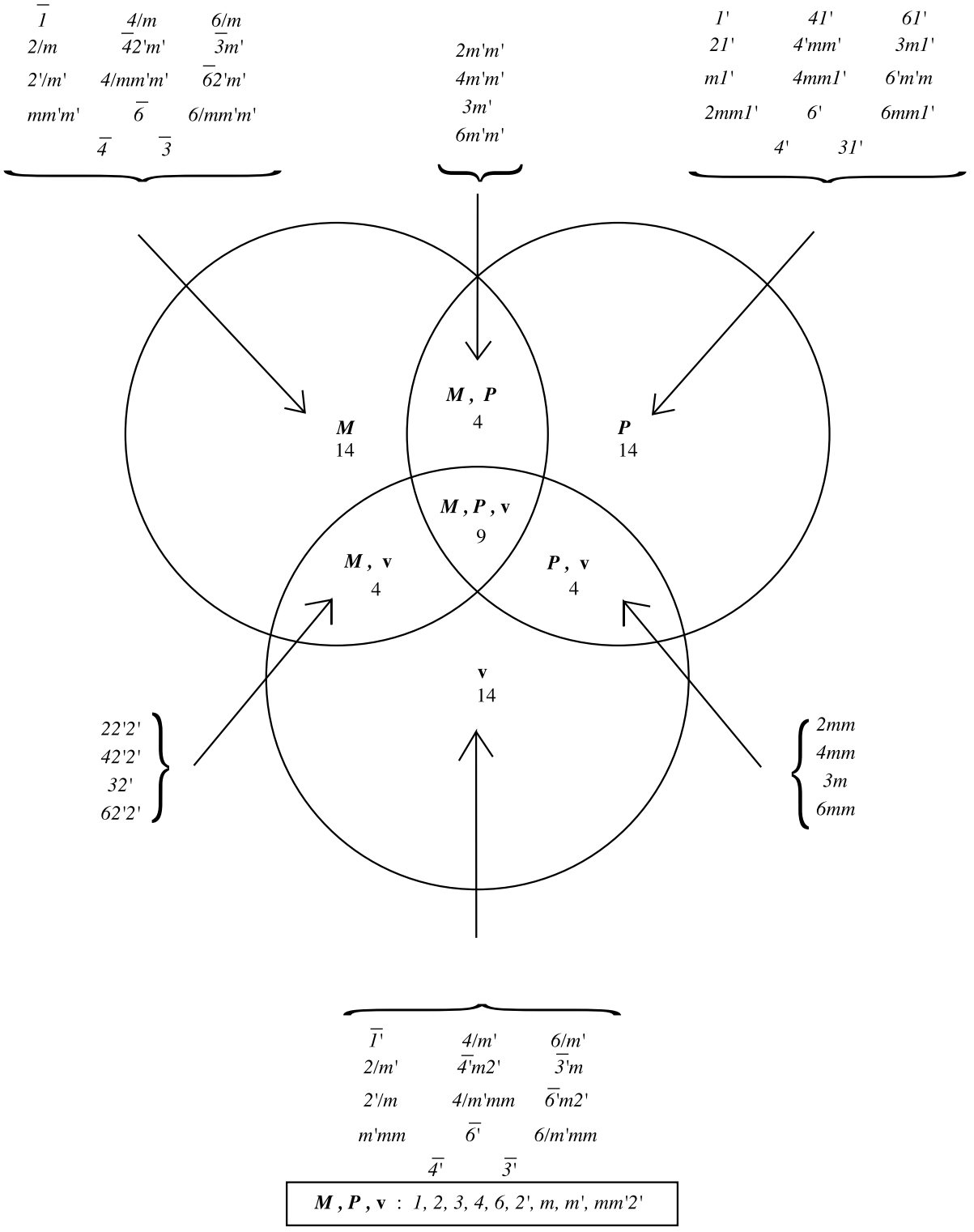
$\bar{1}$ 4/m 6/m
2/m $\bar{4}$2'm' $\bar{3}$m'
2'/m' 4/mm'm' $\bar{6}$2'm'
mm'm' $\bar{6}$ 6/mm'm'
$\bar{4}$ $\bar{3}$
2m'm'
4m'm'
3m'
6m'm'
1' 41' 61'
21' 4'mm' 3m1'
m1' 4mm1' 6'm'm
2mm1' 6' 6mm1'
4' 31'
M , P
4
M
14
P
14
M , P , v
9
M , v
4
P , v
4
v
14
22'2'
42'2'
32'
62'2'
2mm
4mm
3m
6mm
$\bar{1}$' 4/m' 6/m'
2/m' $\bar{4}$'m2' $\bar{3}$'m
2'/m 4/m'mm $\bar{6}$'m2'
m'mm $\bar{6}$' 6/m'mm
$\bar{4}$' $\bar{3}$'
M , P , v : 1, 2, 3, 4, 6, 2', m, m', mm'2'

Figure 1

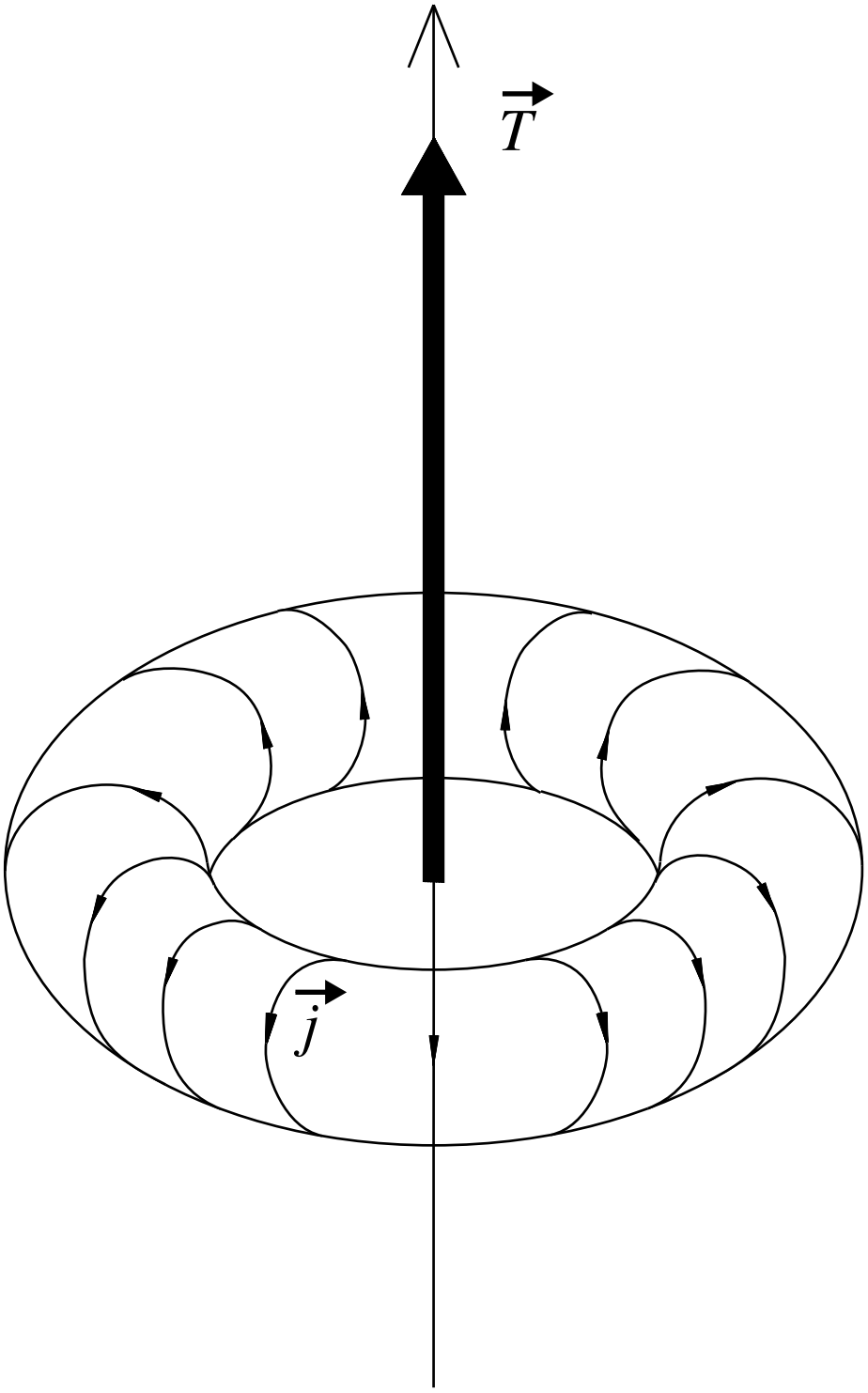

Figure 2

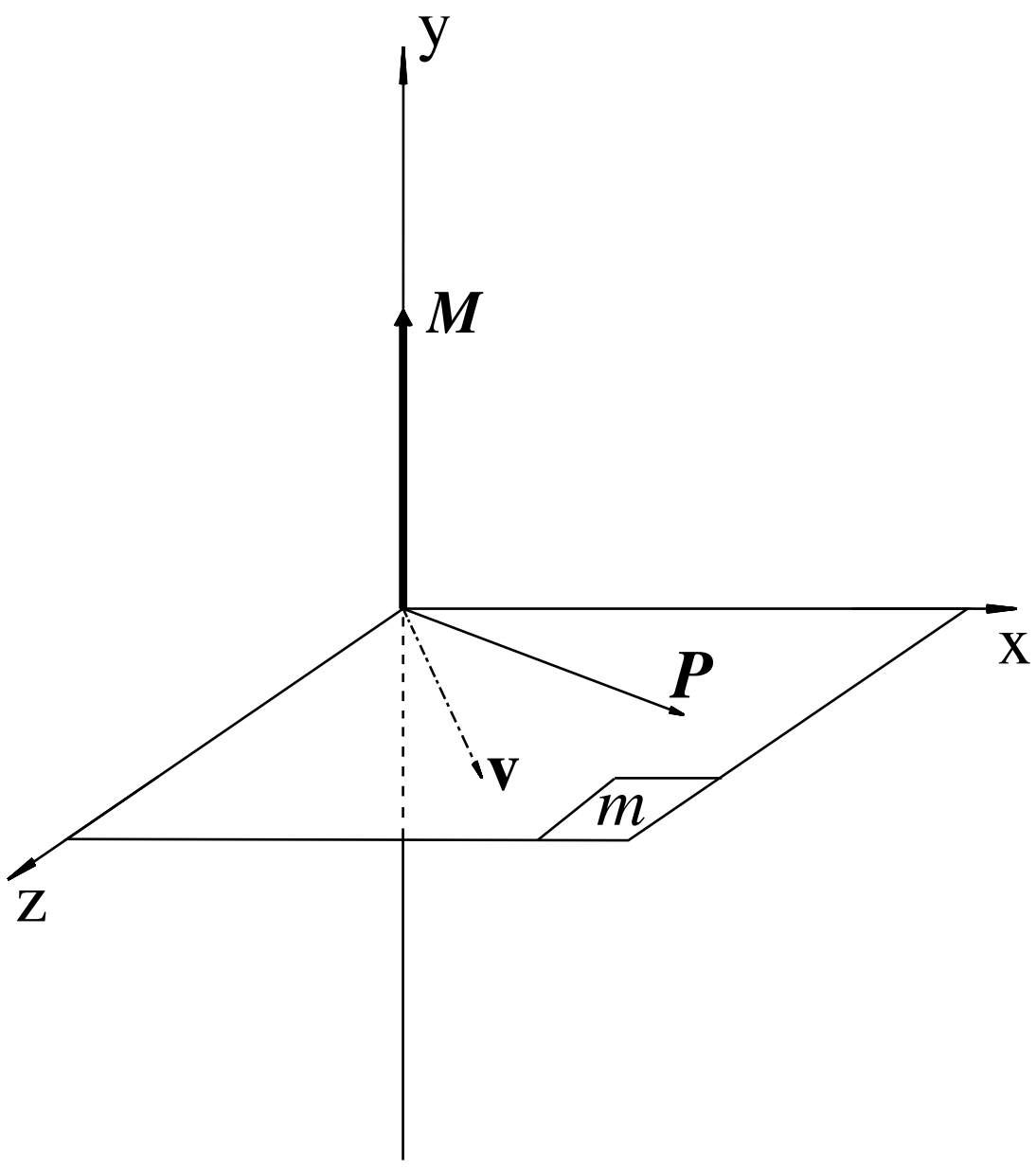

Figure 3